

The Quantitative Characterization of Finite Simple Groups *

In celebration of the 80 birthday of Professor John Thompson

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Abstract

In this report we summarize this work, all finite simple groups G can determined uniformly using their orders $|G|$ and the set $\pi_e(G)$ of their element orders.

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1 Introduction

Group theory is an important branch of mathematics. It has a wide range of applications in other mathematics, physics, chemistry, and other fields. Completed (announced) in 1980, the classification theorem of finite simple groups, is one of the most important mathematical achievements of the 20th century. It is long time to prove this theorem (D. Gorenstein called it the "Thirty Years War"[1]), participants in many countries in hundreds of group theory scientist, many articles (close to 500), length (more than 15,000 pages), is unprecedented in the history of mathematics, with landmark significance.

For a finite group, the order of group and the element order are two of the most important basic concepts. Let G be a finite group and $\pi_e(G)$ be the set of element orders in G . In 1987, the author of this paper posed the following conjecture[2]:

Conjecture. Let G be a group and M a finite simple group. Then $G \cong M$ if and only if (a) $\pi_e(G) = \pi_e(M)$, and (b) $|G| = |M|$.

That is, for all finite simple groups we may characterize them using only their orders and the sets of their element orders (briefly, "two orders").

After I wrote some letters to Prof. John G. Thompson and reported the above conjecture. Thompson pointed that, "Good luck with your conjecture about simple groups. I

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hope you continue to work on it", "This would certainly be a nice theorem", in his reply letters. The warmly encouragement prompt us to finish this work.

Moreover, Thompson posed the following problem and conjecture:

For each finite group G and each integer $d \geq 1$, let $G(d) = \{x \in G; x^d = 1\}$. G_1 and G_2 are of the same order type if and only if $|G_1(d)| = |G_2(d)|, d = 1, 2, \dots$.

Thompson's Problem (1987). Suppose G_1 and G_2 are groups of the same order type. Suppose also G_1 is solvable. Is it true that G_2 is also necessarily solvable?

If G is a finite group, set $N(G) = \{n \in N; G \text{ has a conjugacy class } C \text{ with } |C| = n\}$.

Thompson's Conjecture (1988). If G and M are of finite groups and $N(G) = N(M)$, and if in addition, M is a non-Abelian simple group while the center of G is 1, then G and M are isomorphic.

From 1987 to 2003, the authors of [2-8] proved that this conjecture is correct for all finite simple groups except B_n, C_n and D_n (n even). In the end of 2009, the authors of [9] proved that this conjecture is correct for B_n, C_n and D_n (n even). Thus, this conjecture is proved and become a theorem, that is, **all finite simple groups can determined by their "two orders"**.

Question 1. Find the application for all finite simple groups can determined by their "two orders".

Let G be a finite group and $B(G)$ be Burnside ring of G . We have the following application:

Corollary 1. Let G be a finite simple group. Then $B(G)$ determines G up to isomorphism. **Proof.** See [10, Theorem 5.3].

Question 2. Proving this conjecture, can or not independent on the classification theorem of finite simple groups? For a small number of nonabelian simple group, for example, A_5 , we may do it (see [11]).

Question 3. Weaken the condition of "two orders", characterize all finite simple groups.

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